Journal of Pure and Applied Algebra 35 (1985) 185–196
North-Holland

GRADE FUNCTIONS AND TWO CLASSICAL THEOREMS

L.J. RATLIFF, Jr.*
Department of Mathematics, University of California, Riverside, CA 92521, USA

Communicated by H. Bass
Received 15 September 1983

If R is a Noetherian ring, then for each ideal I in R a finite subset $A(I)$ of Spec R is specified. Then it is shown that if the $A(I)$ satisfy certain natural conditions, then a grade function, Agd(I), can be defined that has many of the basic properties of the classical grade, grade(I). Finally, it is shown that the Agd version of Macaulay's Unmixedness Theorem and of another important theorem concerning Cohen–Macaulay rings hold in this more general situation.

1. Introduction

All rings in this paper are assumed to be commutative Noetherian rings with identity.

In two recent papers, [10] and [8], the concepts of (asymptotic prime divisor, asymptotic sequence, and asymptotic grade) and (essential prime divisor, essential sequence, and essential grade) were introduced and shown to have many of the basic properties known to hold for (prime divisors, R-sequences, and classical grade). In Section 2 of this paper we give an abstract approach to this subject by specifying for each ideal I in a Noetherian ring R a finite subset $A(I)$ of Spec R; the primes in $A(I)$ are the A-prime divisors of I. If the $A(I)$ satisfy certain natural conditions, then we show that the concept of an A-sequence in R and of the A-grade of I can be defined in such a way that many of the basic properties of R-sequences and classical grade have valid analogues in the A-theory. In Section 3 we give several examples (some known and some new) of suitable choices for $A(I)$. In Section 4 we show that the A-theory version of the following classical theorem of Macaulay, [5, (50)], holds for all the examples in Section 3, and also that I^1 is A-grade-unmixed in the general A-theory: If I is an ideal of the principal class in $K[X_1, \dots, X_n]$, then I^k is height-unmixed for all $k \geq 1$. And it is also shown in Section 4 that the A-theory version of four classical corollaries of this theorem hold. Finally, in Section 5 we define an A-ring, briefly discuss some of the properties of such a ring, and then prove the A-theory analogue of the following theorem, [2,

* Research on this paper was partially supported by the National Science Foundation, Grant MCS-8301248.

Theorem 2.2]: If p is a prime ideal in a Noetherian ring R such that R_p is Cohen–Macaulay, then R_P is also Cohen–Macaulay for all but finitely many ideals P in $\{P \in \operatorname{Spec} R;\ \ p \subset P$ and height $P/p = 1\}$.

I am indebted to Steve McAdam for several helpful conversations concerning this paper.

2. A-prime divisors, A-sequences, and A-grade

In this section we define the three terms just mentioned and then derive some of their basic properties. We begin with the definitions.

2.1. Definition. Let R be a Noetherian ring.

(1) For each ideal I in R let $A(I)$ be a subset of $\operatorname{Spec} R$ such that:

(a) $A(I)$ is finite and $I \subseteq P$ for each $P \in A(I)$.

(b) If P is minimal in $\operatorname{Ass} R/I$, then $P \in A(I)$.

(c) If $(x_1, \dots, x_n)R + (y_1, \dots, y_n)R \subseteq P \in \operatorname{Spec} R$ and if $x_i \notin \bigcup A((x_1, \dots, x_{i-1})R)$ and $y_i \notin \bigcup A((y_1, \dots, y_{i-1})R)$ for $i = 1, \dots, n$, then $P \in A((x_1, \dots, x_n)R)$ if and only if $P \in A((y_1, \dots, y_n)R)$.

(2) The ideals in $A(I)$ are called the *A-prime divisors* of I.

(3) Elements $x_1, \dots, x_n$ in R are an *A-sequence* in case $x_i \notin \bigcup A((x_1, \dots, x_{i-1})R)$ for $i = 1, \dots, n$.

(4) The A-grade of I, denoted $\operatorname{Agd}(I)$, is the length of an A-sequence maximal with respect to coming from I.

It is clear that an axiom similar to 2.1(1)(c) is needed in order to have the concept of the A-grade of an ideal well defined. The reason for axiom 2.1(1)(b) is to insure that $\operatorname{Agd}(I) \leq \text{height}\, I$. (For example, if (R, M) is a local ring of altitude $d > 0$ that has a minimal prime ideal $z \neq (0)$ such that $z \notin A((0))$, then $x_1 \in z$, $\notin \bigcup A((0))$, would be an A-sequence of length one and height zero.) Finally, axiom 2.1(1)(a) is included so that the $A(I)$ are somewhat analogous to $\operatorname{Ass} R/I$ (and, in particular, because we want the unions in 2.1(1)(c) and 2.1(3) to be finite unions). However, it is possible that $P \in A(I)$, $\notin \operatorname{Ass} R/I$, so it may seem strange to call such an ideal P an A-prime divisor of I. But in the asymptotic and essential theories (see 3.3 and 3.4) there do exist ideals I such that some of the asymptotic and essential prime divisors of I are not in $\operatorname{Ass} R/I$. (However, in these specific theories, such prime divisors are always in $\operatorname{Ass} R/I^k$ for all large k; but the A-theory axioms do not require even this to hold.)

We now derive some of the consequences of these definitions; all parts of Proposition 2.2 are analogues of known facts concerning R-sequences and classical grade.

2.2. Proposition. *Let R be a Noetherian ring and let $\{A(I);\ I$ is an ideal in $R\}$ be* *a collection of subsets of* Spec R *such that* 2.1(1)(a)–(c) *hold. Then the following hold:*

(1) *The following are equivalent:* (a) $x_1, \dots, x_n$ *are an A-sequence;* (b) $x_1^{k_1}, \dots, x_n^{k_n}$ *are an A-sequence for some positive integers k_i; and,* (c) (b) *holds for all positive integers k_i. If $x_1, \dots, x_n$ are an A-sequence, then* $A((x_1, \dots, x_i)R) = A((x_1^{k_1}, \dots, x_i^{k_i})R)$ *for $i = 1, \dots, n$ and for all positive integers k_i.*

(2) $\mathrm{Agd}(I)$ *is unambigously defined.*

(3) *If $I \subseteq J$, then* $\mathrm{Agd}(I) \le \mathrm{Agd}(J)$.

(4) $\mathrm{Agd}(I) = \mathrm{Agd}(P)$ *for some minimal prime divisor P of I.*

(5) *If $x_1, \dots, x_n$ are an A-sequence, then* $\mathrm{height}(x_1, \dots, x_i)R = i$ *for $i = 1, \dots, n$.*

(6) $\mathrm{Agd}(I) \le \mathrm{height}\, I$.

(7) *If $x_1, \dots, x_n$ are an A-sequence, then* $\mathrm{Agd}((x_1, \dots, x_n)R) = n$.

(8) *If* $\mathrm{Rad}\, I = \mathrm{Rad}\, J$, *then* $\mathrm{Agd}(I) = \mathrm{Agd}(J)$.

Proof. (1) follows immediately from 2.1(1)(c).

For (2) let $x_1, \dots, x_n$ and $y_1, \dots, y_m$ be A-sequences maximal with respect to coming from I and let $m \le n$. Then $I \subseteq P$ for some $P \in A((y_1, \dots, y_m)R)$ and $x_1, \dots, x_m$ are an A-sequence contained in $I \subseteq P$, so $P \in A((x_1, \dots, x_m)R)$, by 2.1(1)(c). Therefore, since $x_1, \dots, x_n$ are an A-sequence contained in P, it follows from 2.1(3) that $n = m$.

(3) is clear.

For (4) let $x_1, \dots, x_n$ be an A-sequence maximal with respect to coming from I, so $I \subseteq Q$ for some $Q \in A((x_1, \dots, x_n)R)$. Then Q contains a minimal prime divisor P of I, so by (3) we have $n = \mathrm{Agd}(I) \le \mathrm{Agd}(P) \le \mathrm{Agd}(Q)$. Also $\mathrm{Agd}(Q) \le n$, by 2.1(3) and 2.1(4), hence $\mathrm{Agd}(I) = \mathrm{Agd}(P)$.

(5) follows readily form 2.1(1)(b) and the Generalized Principal Ideal Theorem.

For (6), if $\mathrm{Agd}(I) = n$ and $x_1, \dots, x_n$ are an A-sequence from I, then it follows from (5) that $\mathrm{Agd}(I) = n = \mathrm{height}(x_1, \dots, x_n)R \le \mathrm{height}\, I$.

For (7), it is clear that $\mathrm{Agd}((x_1, \dots, x_n)R) \ge n$, so the conclusion follows from (5) and (6).

The proof of (8) is similar to the proof of (2), but pick the x's in I and the y's in J. □

In 2.3 we note two results from the standard theory that do not hold in the a-theory.

2.3. Remark. Let (R, M) be a local ring and let $\{A(I);\, I \text{ is an ideal in } R\}$ be a collection of subsets of Spec R such that 2.1(1)(a)–(c) hold. Then the following hold:

(1) A permutation of an A-sequence in R need not be an A-sequence in R.

(2) If I is an ideal in R and $x \in M, \notin I$, then it is possible that $\mathrm{Agd}((I, x)R) > \mathrm{Agd}(I) + 1$.

Proof. As noted in Example 3.1, $A(I) = \{P \in \mathrm{Spec}\, R;\, P \text{ is minimal in } \mathrm{Ass}\, R/I\}$

satisfies axioms 2.1(1)(a)–(c). Therefore let (R, M) be a local ring such that altitude $R=2$ and Ass $R=\{z, w\}$ with z minimal and depth $z=1$. Let x_1 be a regular element in M, so $Q=(z, x_1)R$ is M-primary. Let $y \in Q$, $\notin \bigcup A(x_1R)$, say $y=x_2+rx_1$ with $x_2 \in z$ and $r \in R$. Then $(x_1, x_2)R=(x_1, y)R$ is M-primary, so $x_2 \notin \bigcup A(x_1R)$, and so x_1, x_2 are an A-sequence. However, x_2, x_1 are not an A-sequence by 2.2(5), since $x_2 \in z$. Therefore (1) holds.

For (2), note that height$(z, x_1)R=2>$height $z+1$, and it is readily checked that height is a suitable Agd function for this collection of subsets $A(I)$ of Spec R, so (2) holds. □

Concerning 2.3, it is shown in [10, (2.10)], [8, (4.6)], and [4, (2.5.8)] that a permuation of an A-sequence is again an A-sequence if R is local and if A is as in Examples 3.3–3.9, and it is well known that this also holds with A as in Example 3.2. It is also well known that Agd$((I, x)R) \leq$ Agd$(I)+1$ if $(I, x)R$ is contained in the Jacobson radical of R and Agd is as in Example 3.2, and it is shown in [10, (3.11)], [8, (5.8)], and [4, (2.5.8)] that this also holds with Agd as in Examples 3.3–3.9.

In order to be able to prove the classical theorems for Agd(I) it is necessary to use localizations. Since the definitions in 2.1 depend on the ring R, we now extend them to localizations by:

2.1. (1)(d) If S is a multiplicatively closed set in R and $I_S \neq R_S$, then $A(I_S)=\{P_S; P \in A(I)$ and $P \cap S=\emptyset\}$; and then replace R in 2.1(3) by R_S and I in 2.1(4) by I_S.

(It is readily checked that $\{A(J); J$ is an ideal in $R_S\}$ satisfy 2.1(1)(a)–(c), so the results listed in 2.2 continue to hold for ideals and elements in R_S. This will be implicitely used in what follows.)

With 2.1(1)(d) we can derive three additional properties; as with 2.2, the properties listed in 2.4 are analogues of known facts from the standard theory.

2.4. Proposition. *Let R be a Noetherian ring, let $\{A(I); I$ is an ideal in $R\}$ be a collection of subsets of* Spec R *such that* 2.1(1)(a)–(d) *hold, and let S be a multiplicatively closed set in R. Then the following hold:*

(1) *If $x_1, \ldots, x_n$ are an A-sequence in R and if $(x_1, \ldots, x_n)R_S \neq R_S$, then the images of $x_1, \ldots, x_n$ are an A-sequence in R_S and* Agd$((x_1, \ldots, x_n)R_S)=n$.

(2) *If I is an ideal in R such that $I_S \neq R_S$, then* Agd$(I) \leq$ Agd(I_S).

(3) *If $x_1, \ldots, x_n$ are in the Jacobson radical of R, then $x_1, \ldots, x_n$ are an A-sequence if and only if their images in R_M are an A-sequence for all maximal ideals M in R.*

Proof. (1) follows immediately from 2.1(1)(d) and 2.2(7), and (2) is clear. Finally, (3) follows readily from 2.1(1)(d) and 2.1(3). □

In closing this section we note that 2.1 can be extended to factor rings by: If K is an ideal in R, then $A(I/K)=\{P/K; P\in A(I+K)\}$; and then replace R in 2.1(3) by R/K and I in 2.1(4) by I/K. With this extension it is then straightforward to show that the following are equivalent: (a) $x_1,\dots,x_n$ are an A-sequence; (b) $x_1,\dots,x_i$ are an A-sequence and the images of $x_{i+1},\dots,x_n$ are an A-sequence in $R/(x_1,\dots,x_i)R$ for some $i=1,\dots,n$; and, (c) (b) holds for all $i=1,\dots,n$.

3. Examples

In this brief section we list five known and four new examples of the A-theory developed in Section 2. We begin with the five known examples. 3.1 and 3.2 are, respectively, the coarest and finest; all others lie between these two extreme cases.

3.1. $A(I)=\{P\in \operatorname{Spec} R;\ P$ is minimal in $\operatorname{Ass} R/I\}$. Here, $\operatorname{Agd}(I)=\operatorname{height} I$. It is readily checked that 2.1(1)(a)–(d) hold for these $A(I)$.

3.2. $A(I)=\operatorname{Ass} R/I$ and $\operatorname{Agd}(I)=\operatorname{grade}(I)$. It is well known that 2.1(1)(a)–(d) hold for these $A(I)$.

3.3. $A(I)=\hat{A}^*(I)$; here, $\hat{A}^*(I)=\{P\in \operatorname{Spec} R;\ P$ is a prime divisor of $(I^k)_a$ for all large $k\}$, where $(I^k)_a$ is the integral closure in R of I^k. For this case, $\operatorname{Agd}(I)=\operatorname{agd}(I)$ (called the asymptotic grade of I in [10]) $=\min\{\operatorname{height}(I(R_P)^*+z)/z;\ z$ is a minimal prime ideal in the completion $(R_P)^*$ of R_P and $P\in\hat{A}^*(I)\}$. It is shown in [9, (2.8)] that $\hat{A}^*(I)$ is a finite subset of $\operatorname{Ass} R/I^k$ for all large k, and it is shown in [10] that 2.1(1)(a)–(d) hold.

3.4. $A(I)=E(I)$; here $E(I)=\{P\in \operatorname{Spec} R;$ there exists $z\in\operatorname{Ass}(R_P)^*$ such that $I(R_P)^*+z$ is $P(R_P)^*$-primary$\}$. For this case, $\operatorname{Agd}(I)=\operatorname{egd}(I)$ (called the essential grade of I in [8]) $=\min\{\operatorname{height}(I(R_P)^*+z)/z;\ z\in\operatorname{Ass}(R_P)^*$ and $P\in E(I)\}$. It is shown in [8] that $E(I)$ is a finite subset of $\operatorname{Ass} R/I^k$ for all large k and that 2.1(1)(a)–(d) hold.

3.5. $A(I)=U(I)$; here $U(I)=\{P\in \operatorname{Spec} R;$ there exists $p\in E(\mathrm{u}\mathscr{R}(R,I))$ such that $P=p\cap R\}$, where $\mathscr{R}(R,I)=R[tI,1/t]$ is the Rees ring of R with respect to I and E is as in 3.4. For this case, $\operatorname{Agd}(I)=\operatorname{uegd}(I)$ (called the u-essential grade of I in [4]). It is shown in [4] that $\operatorname{uegd}(I)=\operatorname{egd}(I)$ (see 3.4), that $U(I)$ is a finite subset of $\operatorname{Ass} R/I^k$ for all large k, and that 2.1(1)(a)–(d) hold.

We now mention four new possibilities for $A(I)$. In each case the ideals in $A(I)$ are subsets of $\hat{A}^*(I)\cup E(I)$ (see 3.3 and 3.4), and it was shown in [10] and [8] that 2.1(1)(a)–(d) are satisfied for these ideals. Also, the definition of $\operatorname{Agd}(I)$ is adapted from [10] and [8] where it is verified that it corresponds to the definition in 2.1(4) of $\operatorname{Agd}(I)$.

3.6. Let $A(I)=\{P\in \operatorname{Spec} R$; there exists a minimal $z\in \operatorname{Ass}(R_P)^*$ such that $I(R_P)^*+z$ is $P(R_P)^*$-primary$\}$. Here, $\operatorname{Agd}(I)=\operatorname{agd}(I)$ with $\operatorname{agd}(I)$ as in 3.3.

3.7. $A(I)=\hat{A}^*(I)\cup E(I)$; here $\operatorname{Agd}(I)=\operatorname{egd}(I)$ with $\operatorname{egd}(I)$ as in 3.4.

3.8. If R is a semi-local ring, then let S be a subset of the imbedded prime divisors of zero in R^*, the completion of R. Let $A(I)=\hat{A}^*(I)\cup A_1(I)$, where $\hat{A}^*(I)$ is as in 3.3 and $A_1(I)=\{P^*\cap R$; P^* is a minimal prime divisor if IR^*+z for some some $z\in S\}$. Here, $\operatorname{Agd}(I)=\min\{\operatorname{height}(IR^*+z)/z$; either z is minimal in $\operatorname{Ass} R^*$ or $z\in S\}$.

3.9. Let R and S be as in 3.8 and let $A(I)=\{P^*\cap R$; P^* is a minimal prime divisor of IR^*+z, where either z is minimal in $\operatorname{Ass} R^*$ or $z\in S\}$. Here, $\operatorname{Agd}(I)$ is the same as in 3.8.

3.10. Remark. (1) There is an apparent difference between 3.1–3.2 (where $A(I)\subseteq \operatorname{Ass} R/I$) and 3.3–3.9 (where $A(I)\subseteq \operatorname{Ass} R/I^k$ for all large k). But this difference is easily removed. For it is clear that 3.1 is unaffected if the $A(I)$ there are replaced with $A(I)=\{P\in \operatorname{Spec} R$; P is minimal in $\operatorname{Ass} R/I^k$ for all large $k\}$. And it is shown in [9, (4.1)] that the same Agd function is determined in 3.2 if the $A(I)$ in 3.2 are replaced with $A(I)=\operatorname{Ass} R/I^k$ for all large k. ($\operatorname{Ass} R/I^k$ stabilizes for large k, by [1].)

(2) Some of these examples yield the same Agd function. However, theories with the same Agd function may still have significant differences when considering $\operatorname{Acogd}(I)$, the A-cograde of I. Here, elements $x_1,\dots,x_n$ are an *A-sequence over* an ideal I in case $(I,x_1,\dots,x_n)R\neq R$ and $x_i\notin \bigcup A((I,x_1,\dots,x_{i-1})R)$ for $i=1,\dots,n$, and if R is local, then $\operatorname{Acogd}(I)$ is the maximum length of an A-sequence over I. It is shown in [4] and [11] that there are many differences between $\operatorname{ecogd}(I)$, the essential cograde of I (with $A(I)$ as in 3.4) and $\operatorname{uecogd}(I)$, the u-essential cograde of I (with $A(I)$ as in 3.5), even though 3.4 and 3.5 have the same Agd function.

(3) Somewhat more exotic examples can be given by fixing any of those in 3.1–3.9 and then adjoining to each $A(I)$ an arbitrary finite set of prime ideals containing I, subject only to satisfying 2.1(1)(c) (that is, if $x_1,\dots,x_n$ and $y_1,\dots,y_n$ are A-sequences contained in $P\in \operatorname{Spec} R$, then we put $P\in A((x_1,\dots,x_n)R)$ if and only if we put $P\in A((y_1,\dots,y_n)R)$).

4. Macaulay's unmixedness theorem

As mentioned in the introduction we want to extend Macaulay's Unmixedness Theorem to Agd. Before this, however, we briefly recall some other generalizations of this theorem that have appeared in the literature. In 1957 in [13, Theorem 2.3] Rees captured the essence of Macaulay's Theorem by showing that powers of an ideal generated by an R-sequence in a Noetherian ring are grade-unmixed. (And, as a consequence, [13, Theorem 3.2]: Powers of ideals of the principal class in Cohen–Macaulay rings are height-unmixed.) In 1981 in [10, (3.7)] it was shown that the integral closures of powers of an ideal generated by an asymptotic-sequence in a Noetherian ring are asymptotic-grade-unmixed. (And, therefore, [10, (4.1)]: The integral closures of powers of ideals of the principal class in locally quasi-unmixed Noetherian rings are height-unmixed.) Finally, in 1983 in [8, (5.10)] it was shown that the essential prime divisors of powers of an ideal generated by an essential-sequence in a Noetherian ring all have the same essential-grade. (So, [8, (6.4)]: The essential prime divisors of powers of ideals of the principal class in locally unmixed Noetherian rings are height-unmixed.) Thus it might be hoped that the A-prime divisors of powers of an ideal generated by an A-sequence in a Noetherian ring all have the same A-grade. Unfortunately, this does not hold in general, as we show in 4.1. But we show in 4.2 that this holds for the first power of such ideals, and in 4.3 we show that it does hold for all the examples in Section 3. (Concerning the theorems mentioned in the parentheses, see 5.3.)

4.1. Example. *There exists a local domain (R,M) that has an A-sequence x_1, x_2 such that powers of $(x_1,x_2)R$ are not Agd-unmixed.*

Proof. Let (R,M) be a regular local ring of altitude three and let $A(I) = \operatorname{Ass} R/I$ for all ideals I in R. Therefore $\operatorname{Agd}(M) = 3$, where Agd = grade. Fix a regular system of parameters x_1, x_2, x_3, let $I = (x_1,x_2)R$, and by 3.10(3) redefine $A(I^k)$ for $k \geq 2$ by $A(I^k) = A(I^k) \cup \{M\}$. Then x_1, x_2 are an A-sequence and I^k is not agd-unmixed, since M is an A-prime divisor of I^k and $\operatorname{Agd}(M) = 3$. □

However, at least the following A-theory version of the unmixedness theorem holds.

4.2. Theorem. *Let R be a Noetherian ring and let $\{A(I);\ I$ is an ideal in $R\}$ be a collection of subset of* Spec R *such that* 2.1(1)(a)–(d) *hold. Then ideals generated by A-sequences are Agd unmixed; that is, if X is generated by an A-sequence and P is an A-prime divisor of X, then* $\operatorname{Agd}(P_P) = \operatorname{Agd}(P) = \operatorname{Agd}(X)$.

Proof. Let X be generated by the A-sequence $x_1, \ldots, x_n$ and let P be an A-prime divisor of X. Then $n = \operatorname{Agd}(X)$, by 2.2(7) and $X \subseteq P$, by 2.1(1)(a), so by 2.2(3) and 2.4(2) we have $n = \operatorname{Agd}(X) \leq \operatorname{Agd}(P) \leq \operatorname{Agd}(P_P)$. Also, $P \in A(X)$, so $P_P \in A(X_P)$, by 2.1(1)(d). Further, the images of $x_1, \ldots, x_n$ are an A-sequence in R_P, by 2.4(1), so $\operatorname{Agd}(P_P) = n$, by 2.1(4), hence $n = \operatorname{Agd}(X) = \operatorname{Agd}(P) = \operatorname{Agd}(P_P)$. □

We next show that the A-theory version of Macaulay's Theorem does hold for the examples in Section 3.

4.3. Corollary. *In Examples* 3.1–3.9, *if X is an ideal generated by an A-sequence and P is an A-prime divisor of X^k for some $k \geq 1$, then* $\mathrm{Agd}(P_P) = \mathrm{Agd}(P) = \mathrm{Agd}(X)$.

Proof. If P is an A-prime divisor of X^k, then $P \in A(X^k)$, so $P \in A(X)$: for 3.1 and 3.2, this follows from 3.10(1); for 3.3 and 3.4 it is clear, so it is also clear for 3.6–3.9, since in these examples $A(I) \subseteq \hat{A}^*(I) \cup E(I)$; and for 3.5 it is shown in [4]. Therefore, since 2.1(1)(a)–(d) hold in these examples, the conclusion follows from 4.2. □

We now prove four additional corollaries of 4.2; it is assumed in these corollaries that $\{A(I)\}$ satisfy 2.1(1)(a)–(d). Prior to each of these corollaries we state the classical version.

In [14, Theorem 1.3] Rees proved: If P is a prime ideal in a Noetherian ring R that contains an ideal X generated by an R-sequence of length n, then:

(a) $P \in \mathrm{Ass}\, R/X$ if and only if $\mathrm{grade}(P) = \mathrm{grade}(P_P) = n$; and,

(b) $P \notin \mathrm{Ass}\, R/X$ if and only if $\mathrm{grade}(P_P) > n$.

(4.4) is the A-theory version of this.

4.4. Corollary. *Let P be a prime ideal in a Noetherian ring R that contains an ideal X generated by an A-sequence of length n. Then the following hold:*

(1) $P \in A(X)$ *if and only if* $\mathrm{Agd}(P) = \mathrm{Agd}(P_P) = n$.

(2) $P \notin A(X)$ *if and only if* $\mathrm{Agd}(P_P) > n$.

Proof. (1) If $P \in A(X)$, then P is an A-prime divisor of X, by definition, so $\mathrm{Agd}(P) = \mathrm{Agd}(P_P) = n$, by 4.2. And, if $\mathrm{Agd}(P_P) = n$, then it follows from hypothesis and 2.4(1) that $\mathrm{Agd}(X_P) = n = \mathrm{Agd}(P_P)$, so $P_P \in A(X_P)$, by 2.1(4), hence $P \in A(X)$, by 2.1(1)(d).

(2) follows immediately from (1) and 2.4(1). □

Concerning 4.4, it is well known in the standard theory that to have $P \in \mathrm{Ass}\, R/X$ it is not sufficient for $\mathrm{grade}(P) = \mathrm{grade}(X)$, so to have $P \in A(X)$ it is not sufficient for $\mathrm{Agd}(P) = \mathrm{Agd}(X)$. Similarly, it is possible for $P \in A(I) \cap A(J)$ for some ideals I and J in R such that $\mathrm{Agd}(I) \neq \mathrm{Agd}(J)$, but this cannot happen if I and J are generated by A-sequences (of different length), since otherwise P_P would contain maximnal A-sequences of different length.

In [14, Corollary, p. 183] Rees proved: If X and Y are ideals generated by R-sequences of length n and if $\mathrm{Rad}\, X = \mathrm{Rad}\, Y$, then $\mathrm{Ass}\, R/X = \mathrm{Ass}\, R/Y$. 4.5 is the A-theory version of this.

4.5. Corollary. *If X and Y are ideals in a Noetherian ring R generated by A-sequences of length n, and if* $\mathrm{Rad}\, X = \mathrm{Rad}\, Y$, *then* $A(X) = A(Y)$.

Proof. If $P \in A(X)$, then $\mathrm{Agd}(P_P)=n$, by 4.2. Also $P_P \in A(X_P)$, by 2.1(1)(d), and X_P and Y_P are generated by A-sequences of length n, by 2.4(1). Therefore $P_P \in A(Y_P)$, by 2.1(1)(c), hence $P \in A(Y)$, by 2.1(1)(d). Thus $A(X) \subseteq A(Y)$, so they are equal by symmetry. □

In [12, Theorem 3.5] Rees proved:

(a) If X is an ideal generated by an R-sequence of length n, then a minimal (maximal) prime divisor of X is a minimal (maximal) prime ideal of grade n; and,

(b) If P is a minimal (maximal) prime ideal of grade n and X is an ideal generated by an R-sequence of length n contained in P, then P is a minimal (maximal) prime divisor of X.

4.6 is the A-theory analogue of this.

4.6. Corollary. *Let R be a Noetherian ring, let $P \in \operatorname{Spec} R$, let $X \subseteq P$ be an ideal generated by an A-sequence of length n, and let $G=\{P \in \operatorname{Spec} R;\ \mathrm{Agd}(P)=n\}$. Then the following hold:*

(1) *If P is minimal (maximal) in $A(X)$, then P is minimal (maximal) in G.*

(2) *If P is minimal (maximal) in G and $X \subseteq P$, then P is minimal (maximal) in $A(X)$.*

Proof. (1) If $P \in A(X)$, then $\mathrm{Agd}(P)=n$, by 4.2, so $P \in G$. Now, if P is minimal in $A(X)$ and $Q \in \operatorname{Spec} R$, then $Q \subset P$ implies that $\mathrm{Agd}(Q) \leq \text{height } Q$, by 2.2(6), and height $Q <$ height $P \leq n$, by the Generalized Principal Ideal Theorem, so P is minimal in G. And if P is maximal in $A(X)$ and $P \subset Q \in \operatorname{Spec} R$, then Q is not an A-prime divisor of X. Therefore, since $X \subseteq Q$, it follows from the definition that $\mathrm{Agd}(Q)>n$, and so P is maximal in G.

(2) If P is minimal in G and $X \subseteq P$, then P contains a minimal prime divisor p of X. Then $n=\mathrm{Agd}(X) \leq \mathrm{Agd}(p) \leq \mathrm{Agd}(P)=n$, so P is minimal in $A(X)$ by the hypothesis. Also, if P is maximal in G and $X \subseteq P$, then $\mathrm{Agd}(P)=n=\mathrm{Agd}(X)$ implies that $P \subseteq Q$ for some $Q \in A(X)$. Then $\mathrm{Agd}(Q)=n$, by 4.2, and so the hypothesis implies that P is maximal in $A(X)$. □

The final corollary of 4.2 gives a nice application of 4.6. Its standard theory version was proved by Rees in [12, Corollary, p. 610]: If I is an ideal of grade n in a Noetherian ring R, then there are at most finitely many maximal ideals of grade n containing I.

4.7. Corollary. *If I if an ideal in a Noetherian ring R such that $\mathrm{Agd}(I)=n$, then there are at most finitely many maximal ideals M in R containing I such that $\mathrm{Agd}(M)=n$.*

Proof. Let X be an ideal generated by an A-sequence of length n contained in I. Then if M is a maximal ideal in R such that $I \subseteq M$ and $\mathrm{Agd}(M)=n$, then $M \in A(X)$, by 4.6(2), so there are only finitely many such M, by 2.1(1)(a). □

4.8. Remark. Since Examples 3.1–3.9 satisfy 2.1(1)(a)–(d), it follows that 4.4–4.7 hold for these A-theories. Of course, this was already known for Example 3.2 (the standard theory), but the preceding gives a new proof of this. And 4.3–4.7 are new results for the other theories, except 4.3 was shown for Example 3.4 in [8, (5.10)]. (4.3 is new for Example 3.3, since we are only considering X^k and not $(X^k)_a$.)

5. The second theorem

In this section we first define A-rings and briefly discuss some of their properties, and we then prove the A-theory version of [2, Theorem 2.2]; this theorem is certainly not as classical as Macaulay's Theorem, but it is a useful and important result.

We begin with the definition.

5.1. Definition. An *A-ring* is a Noetherian ring R in which $\mathrm{Agd}(M) = \mathrm{height}\, M$ for all maximal ideals M in R.

5.1 is adapted from [3, p. 95] where Kaplansky defines Cohen–Macaulay rings in the analogous manner. And it follows from [10, (4.1)] and [8, (6.1)] that locally quasi-unmixed Noetherian rings (asymptotic theory) and locally unmixed Noetherian rings (essential theory) could also be defined in a similar manner.

For the examples in Section 3, the A-rings for 3.1 are all Noetherian rings, for 3.2 they are the Cohen–Macaulay rings, for 3.3 and 3.6 they are the locally quasi-unmixed Noetherian rings; and for 3.4, 3.5, 3.7–3.9 they are the locally unmixed Noetherian rings.

We now give a corollary of the definition.

5.2. Corollary. *Let R be a Noetherian ring and let $\{A(I);\ I$ is an ideal in $R\}$ be a collection of subsets of* $\mathrm{Spec}\, R$ *such that* 2.1(1)(a)–(d) *hold. If X is an ideal generated by an A-sequence $x_1, \dots, x_n$ and if P is a minimal prime divisor of X, then R_P is an A-ring.*

Proof. It follows from 2.2(7) and the Generalized Principal Ideal Theorem that $\mathrm{height}\, P = n$. Also, $\mathrm{Agd}(P_P) = n$, by 2.1(1)(b) and 4.2, and $\mathrm{height}\, P_P = \mathrm{height}\, P = n$, so the conclusion is clear by 5.1.

5.3. Remark. (1) Consider the following statements:

(a) Localizations of an A-ring are A-rings;

(b) Agd and height coincide in A-rings;

(c) If $x_1, \dots, x_n$ are an A-sequence in an A-ring R and $P \in A((x_1, \dots, x_n)R)$, then $\mathrm{height}\, P = n$; and,

(d) If $x_1, \dots, x_n$ are a system of parameters in a local A-ring, then $x_1, \dots, x_n$ are an A-sequence.

Then it is straightforward to show that if 2.1(1)(a)–(d) hold, then (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c). (Concerning (d), see Remark (2).)

(2) If it is only assumed that 2.1(1)(a)–(d) hold, then none of (1)(a)–(d) hold for all A-rings. For example, let $(R, M=(x_1, x_2)R)$ be a regular local ring of altitude two and let $A(I) = \operatorname{Ass} R/I$ for all ideals I in R. By 3.10(3) redefine $A((0))$ by $A((0)) = \{(0), x_1R\}$. Then R is an A-ring, since x_1+x_2, x_2 are an A-sequence of length two in M, but height $x_1R=1$ and $\operatorname{Agd}(x_1R)=0$. Therefore (1)(c) does not hold, so neither do (1)(a) nor (1)(b), by (1). And this same example, using x_1, x_2, shows that (1)(d) also does not hold.

(3) If we adopt the following axiom 2.1(1)(e) (together with 2.1(1)(a)–(d), then it is straightforward to show that (1)(b) holds, so (1)(a) and (1)(c) also hold, by (1). It is also straightforward to show that height P + depth P = altitude R for all prime ideals P in R, when R is local, so it follows that (1)(d) also holds.

2.1. (1)(e) If $x_1, \ldots, x_n$ $(n \geq 0)$ are an A-sequence in a Noetherian ring R, if $x_{n+1} \notin \bigcup A((x_1, \ldots, x_n)R)$, and if $P \in A((x_1, \ldots, x_n)R)$ is such that $(P, x_{n+1})R \neq R$, then each minimal prime divisor of $(P, x_{n+1})R$ is in $A((x_1, \ldots, x_{n+1})R)$.

(It is shown in [7, (3.2)] together with [4, (5.1) and (2.5.8)] that 2.1(1)(e) holds for Examples 3.3–3.9, and it is well known that it holds for 3.1 and 3.2, so 5.3(1)(a)–(d) hold for all these examples, by 5.3(3).)

5.3(2) showed that a localization of an A-ring may not be an A-ring. However, we show in 5.5 that the following theorem from the standard theory, [2, Theorem 2.2], does generalize by assuming only 2.1(1)(a)–(d): If p is a prime ideal in a Noetherian ring R such that R_p is a Cohen–Macaulay ring, then R_P is a Cohen–Macaulay ring for all but finitely many $P \in \{P \in \operatorname{Spec} R; p \subset P$ and height $P/p = 1\}$. (The asymptotic version of this result is given in [6, Chapter V], and the essential version is given in [8, (6.5)].)

To generalize this result we will use the following lemma from [2].

5.4. Lemma [2, Lemma 2.1]. *Let p be a prime ideal in a Noetherian ring R, let $X_0, \ldots, X_n$ be ideals in R which are contained in p, and let $\mathcal{P} = \{P \in \operatorname{Spec} R; p \subset P$ and* height $P/p=1\}$. *Then there are at most finitely many $P \in \mathcal{P}$ such that either P contains an ideal $q \in \bigcup_0^n \operatorname{Ass} R/X_j$ such that $q \not\subseteq p$ or* height $P >$ height $p+1$.

5.5. Theorem. *Let R be a Noetherian ring and let $\{A(I)$; I is an ideal in $R\}$ be a collection of subsets of* Spec R *such that* 2.1(1)(a)–(d) *hold. Let $p \in \operatorname{Spec} R$ such that R_p is an A-ring and let $\mathcal{P} = \{P \in \operatorname{Spec} R$; $p \subset P$ and* height $P/p=1\}$. *Then there are at most finitely many $P \in \mathcal{P}$ such that R_P is not an A-ring.*

Proof. Let $x_1, \ldots, x_n$ be elements in p such that their images in R_p are a maximal A-sequence (so height $p=n$) and such that height $X_j = j$ $(j=0,1,\ldots,n)$, where $X_j = (x_1, \ldots, x_j)R$. Then $\bigcup_0^n A(X_j)$ is a finite set of prime ideals in R, by 2.1(1)(a), so by 5.4 only finitely many ideals in $\mathcal{P}$ can contain an element in this set which is not contained in p or can have height greater than $n+1$. Omitting these, it is readily seen that R_P is an A-ring for all other $P \in \mathscr{P}$. □

References

[1] M. Brodmann, Asymptotic stability of Ass(M/I^nM), Proc. Amer. Math. Soc. 74 (1979) 16–18.

[2] M. Hochster and L.J. Ratliff, Jr., Five theorems on Macaulay rings, Pacific J. Math. 44 (1973) 147–172.

[3] I. Kaplansky, Commutative Rings (Allyn and Bacon, Boston, 1970).

[4] D. Katz and L.J. Ratliff, Jr., Essential sequences over an ideal and essential cograde, 45 page preprint.

[5] F.S. Macaulay, Algebraic Theory of Modular Systems, Cambridge Tracts in Math. 19 (Cambridge Univ. Press, Cambridge, 1916).

[6] S. McAdam, Asymptotic Prime Divisors, Lecture Notes in Math. (Springer, Berlin),

[7] S. McAdam and L.J. Ratliff, Jr., On the asymptotic cograde of an ideal, J. Algebra 87 (1984) 36–52.

[8] S. McAdam and L.J. Ratliff, Jr., Essential sequences, J. Algebra, to appear.

[9] L.J. Ratliff, Jr., Asymptotic prime divisors, Pacific J. Math., 111 (1984) 395–413.

[10] L.J. Ratliff, Jr., Asymptotic sequences, J. Algebra 85 (1983) 337–360.

[11] L.J. Ratliff, Jr., Essential sequences over an ideal and essential cograde, Math. Z., to appear.

[12] D. Rees, A theorem of homological algebra, Math. Proc. Cambridge Philos. Soc. 52 (1956) 605–610.

[13] D. Rees, The grade of an ideal or module, Math. Proc. Cambridge Philos. Soc. 53 (1957) 28–42.

[14] D. Rees, A note on general ideals, J. London Math. Soc. 32 (1957) 181–186.